AN IRRESISTIBLE INTEGRAL

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Boros and Moll's classic book *Irresistible Integrals* contains a peculiar but beautiful formula of which we will outline a proof below. The result reads

$$\int_1^\infty \frac{\{x\} - \frac{1}{2}}{x} dx = -1 + \ln\left(\sqrt{2\pi}\right) \approx -0.08106...$$

Here $\{x\}$ denotes the fractional part of x.

To prove the result, one needs Stirling's asymptotic formula for $n!$:

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

Stirling's formula is an asymptotic result which means that as $n \to \infty$, the relative error converges to zero. More precisely, we have

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} = \sqrt{2\pi}$$

We will come back to this formula in due time. To begin to prove the main result, write:

$$\ln\{n!\} = \ln\left(\prod_{k=1}^{n} k\right) = \sum_{k=2}^{\infty} \ln(k)$$

Since

$$\ln(k) = \int_1^k \frac{dx}{x}$$

we have

$$\ln\{n!\} = \sum_{k=2}^{n} \int_1^k \frac{dx}{x}$$

Furthermore, if we subdivide the region of integration, we have

$$\int_1^k \frac{dx}{x} = \sum_{j=1}^{k-1} \int_j^{j+1} \frac{dx}{x}$$

So finally, we obtain

$$\ln\{n!\} = \sum_{k=2}^{n} \left\{ \sum_{j=1}^{k-1} \int_j^{j+1} \frac{dx}{x} \right\}$$

We can now exchange the order of summation. To see why this is the case, let us write the first few values of k for the first sum:

$$k = 2 \Rightarrow \int_1^2$$

$$k = 3 \Rightarrow \int_1^2 + \int_2^3$$

$$k = 4 \Rightarrow \int_1^2 + \int_2^3 + \int_3^4$$

and in general,

$$k = n \Rightarrow \int_1^2 + \int_2^3 + \int_3^4 + \cdots + \int_{n-1}^n$$

Then in the second sum we sum all of these horizontally. This is equivalent to just performing the sum first vertically. In the end, we obtain:

$$\begin{aligned}\ln\{n!\} &= (n-1)\int_1^2 \frac{dx}{x} + (n-2)\int_2^3 \frac{dx}{x} + (n-3)\int_3^4 \frac{dx}{x} + \cdots + \int_{n-1}^n \frac{dx}{x} \\ &= \int_1^2 \frac{(n-1)}{x}dx + \int_2^3 \frac{(n-2)}{x}dx + \int_3^4 \frac{(n-3)}{x}dx + \cdots + \int_{n-1}^n \frac{1}{x}dx\end{aligned}$$

With $1 \le j \le n-1$, each term in the last sum is of the form

$$\int_j^{j+1} \frac{(n-j)}{x}dx = \int_j^{j+1} \frac{n - \lfloor x \rfloor}{x}dx$$

because for the integration interval $j \le x < j+1$, we have $\lfloor x \rfloor = j$. Thus

$$\ln\{n!\} = \int_1^n \frac{n - \lfloor x \rfloor}{x}dx$$

Since $\lfloor x \rfloor = x - \{x\}$, we have $n - \lfloor x \rfloor = n - x + \{x\}$. Plugging this in above yields

$$\begin{aligned}\ln\{n!\} &= \int_1^n \frac{n - x + \{x\}}{x}dx = \int_1^n \frac{n}{x}dx - \int_1^n dx + \int_1^n \frac{\{x\}}{x}dx \\ &= n\ln(n) - (n-1) + \int_1^n \frac{\{x\}}{x}dx \\ &= n\ln(n) - n + 1 + \frac{1}{2}\ln(n) + \int_1^n \frac{\{x\} - \frac{1}{2}}{x}dx \\ &= \left(n + \frac{1}{2}\right)\ln(n) - n + 1 + \int_1^n \frac{\{x\} - \frac{1}{2}}{x}dx \\ &= \ln\left(n^{n+\frac{1}{2}}\right) + \ln(e^{-n}) + \ln\left(e^{1+\int_1^n \frac{\{x\}-\frac{1}{2}}{x}dx}\right)\end{aligned}$$

Therefore

$$\begin{aligned}n! &= n^{n+\frac{1}{2}} \cdot e^{-n} \cdot e^{1+\int_1^n \frac{\{x\}-\frac{1}{2}}{x}dx} \\ \Rightarrow e^{1+\int_1^n \frac{\{x\}-\frac{1}{2}}{x}dx} &= \frac{n!}{n^{n+\frac{1}{2}} \cdot e^{-n}}\end{aligned}$$

Now if we take the limit as $n \to \infty$, we can apply Stirling's formula and we obtain

$$e^{1+\int_1^n \frac{\{x\}-\frac{1}{2}}{x}dx} = \sqrt{2\pi}$$

so that the result follows.