DALZELL'S INTEGRAL

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In 1944, D.P. Dalzell (1898-1988) published a paper in the *Journal of the London Mathematical Society* in which he used an integral to give a nice decimal approximation to π. In this post, we outline his simple but elegant method.

Consider the integral

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$$

After some algebra, we have

$$I = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

Performing the integration gives

$$I = \left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1}(x) \right) \Bigg|_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi$$

This gives us the nice result:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi$$

Since the integrand is always positive, we have $\frac{22}{7} > \pi$. Here are some other easy estimates. Since $1 < 1 + x^2 < 2$ for $0 < x < 1$

$$\int_0^1 \frac{x^4(1-x)^4}{2}dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi < \int_0^1 x^4(1-x)^4 dx$$

Evaluating these integrals gives:

$$\frac{1}{2}\left(\frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9} \right) = \frac{1}{1260} < \frac{22}{7} - \pi < \left(\frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9} \right) = \frac{1}{630}$$

Multiplying the inequality by -1 yields:

$$-\frac{1}{630} < \pi - \frac{22}{7} < -\frac{1}{1260}$$

So that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

This gives us an upper and lower bound on π to five decimal places:

$$3.14127 < \pi < 3.14206$$