BERNOULLI'S INTEGRAL

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In 1697, John Bernoulli evaluated the integral

$$\int_0^1 x^x dx$$

We examine how to compute this integral, as well as some other similar ones:

$$\int_0^1 x^{-x} dx$$

$$\int_0^1 x^{x^2} dx$$

$$\int_0^1 x^{\sqrt{x}} dx$$

To find expressions for these integrals, we will need to use the Gamma function.

Euler's Gamma Function

In two letters in 1730, Leonard Euler created the gamma function, $\Gamma(n)$. Legendre proposed an integral definition for this function:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

We can easily compute the gamma function for $n = 1$:

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

Using integration by parts, we have

$$\Gamma(n+1) = n \int_0^\infty e^{-x} x^{n-1} dx$$

This gives us the functional equation for the gamma function

$$\Gamma(n+1) = n\Gamma(n)$$

In particular, we have, for $n \in \mathbb{Z}^+$,

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2!$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3!$$

and so on. In general:

$$\Gamma(n+1) = n!$$

$$\Rightarrow \Gamma(n) = (n-1)!, \quad n \geq 1$$

This intimate connection between the gamma function and the factorial function was in fact Euler's original motivation for studying $\Gamma(n)$. The gamma function can also be extended for all real and complex arguments - but this is best saved for a discussion some other time. Our particular interest at the moment is to use the gamma function to compute integrals. Here is a simple example of how it can be used in the context of integration.

Example. Find the value of

$$\int_0^\infty e^{-x^3} dx$$

By making the change of variable $y = x^3$, we obtain:

$$\int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-y} \frac{dy}{3y^{\frac{2}{3}}} = \frac{1}{3}\int_0^\infty e^{-y} y^{-\frac{2}{3}} dy$$

This is simply the definition of the gamma function with $n - 1 = -\frac{2}{3} \Rightarrow n = \frac{1}{3}$. So we have

$$\int_0^\infty e^{-x^3} dx = \frac{1}{3}\Gamma(\frac{1}{3})$$

A Lemma

To compute the integrals above, we will need the following lemma.

Lemma.

$$\int_0^1 x^m \ln^n(x) dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Proof. Make the change of variable $u = -\ln(x) \Rightarrow x = e^{-u}$. Then we have

$$\int_0^1 x^m \ln^n(x) dx = \int_\infty^0 e^{-um}(-u)^n(-e^{-u}) du = (-1)^n \int_0^\infty u^n e^{-(m+1)u} du$$

Now let $t = (m+1)u$, so that

$$(-1)^n \int_0^\infty u^n e^{-(m+1)u} du = (-1)^n \int_0^\infty \left(\frac{t}{m+1}\right)^n e^{-t} \frac{dt}{m+1}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty t^n e^{-t} dt$$

But this last integral is simply $\Gamma(n+1)$ □

Bernoulli's Integral

Now we are ready to evaluate Bernoulli's integral. We begin with the identity

$$x^{cx^a} = e^{cx^a \ln(x)}$$

where a and c are constants. Then, since $e^x = 1 + x + \frac{x^2}{2!} + ...$, we have

$$x^{cx^a} = 1 + cx^a \ln(x) + \frac{1}{2!}c^2 x^{2a} \ln^2(x) + \frac{1}{3!}c^3 x^{3a} \ln^3(x) + \cdots$$

so that

$$\int_0^1 x^{cx^a} dx = \int_0^1 dx + c\int_0^1 x^a \ln(x) dx + \frac{c^2}{2!}\int_0^1 x^{2a} \ln^2(x) dx + \frac{c^3}{3!}\int_0^1 x^{3a} \ln^3(x) dx + ...$$

Now each of these integrals can be evaluated using the Lemma above:

$$\int_0^1 x^{cx^a} dx = 1 - \frac{c}{(a+1)^2} + \frac{c^2}{2!}\left\{\frac{2!}{(2a+1)^3}\right\} - \frac{c^3}{3!}\left\{\frac{3!}{(3a+1)^4}\right\} + \frac{c^4}{4!}\left\{\frac{4!}{(4a+1)^5}\right\} - ...$$

$$\int_0^1 x^{cx^a}dx = 1 - \frac{c}{(a+1)^2} + \frac{c^2}{(2a+1)^3} - \frac{c^3}{(3a+1)^4} + \frac{c^4}{(4a+1)^5} - ...$$

With $c = a = 1$, we have Bernoulli's integral:

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots \approx 0.78343...$$

With $c = -1, a = 1$, we have:

$$\int_0^1 x^{-x} dx = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} - \cdots \approx 1.29128...$$

Incidentally, this can be written to give this remarkable formula:

$$\int_0^1 \frac{1}{x^x}dx = \sum_{k=1}^{\infty} \frac{1}{k^k}$$

With $c = 1, a = 2$, we have:

$$\int_0^1 x^{x^2} dx = 1 - \frac{1}{3^2} + \frac{1}{5^3} - \frac{1}{7^4} + \frac{1}{9^5} - \cdots \approx 0.896488...$$

Finally, with $c = 1, a = \frac{1}{2}$, we have:

$$\int_0^1 x^{\sqrt{x}} dx = 1 - \frac{1}{\left(\frac{3}{2}\right)^2} + \frac{1}{\left(\frac{4}{2}\right)^3} - \frac{1}{\left(\frac{5}{2}\right)^4} + \frac{1}{\left(\frac{6}{2}\right)^5} - \cdots \approx 0.658582...$$

Collecting all of our formulas, we have the following lovely results:

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots \approx 0.78343...$$

$$\int_0^1 x^{-x} dx = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} - \cdots \approx 1.29128...$$

$$\int_0^1 x^{x^2} dx = 1 - \frac{1}{3^2} + \frac{1}{5^3} - \frac{1}{7^4} + \frac{1}{9^5} - \cdots \approx 0.896488...$$

$$\int_0^1 x^{\sqrt{x}} dx = 1 - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{4}\right)^3 - \left(\frac{2}{5}\right)^4 + \left(\frac{2}{6}\right)^5 - \cdots \approx 0.658582...$$

By the way, Bernoulli was so fascinated by this beautiful result that he called it his "series mirabili" ("marvelous series"). I couldn't agree with him more.